I. EVIDENCE FOR DARK MATTER

Since its discovery in the 1930s, the evidence for dark matter has been plentiful. Dark matter was first discovered by observations that rotation curves in galaxies and clusters differed significantly from the rotation curve expected from the observed luminous matter [1]. The luminous matter of galaxies is largely contained within the inner few kiloparsecs of the galaxy. According to the law of gravitation, in the outer regions of a galaxy, the circular velocity of observed bodies should therefore follow:

\[ F_G = \frac{G M(< r)}{r^2} = \frac{m v_{\text{cir}}^2}{r}, \tag{1.1} \]

\[ v_{\text{cir}}(r) \propto r^{-1/2}, \tag{1.2} \]

with \( M(< r) \) the mass contained with radius from the galactic center \( r \) and \( v_{\text{cir}} \) the circular velocity at radius \( r \). However, the circular velocity tends to flatten out at large radii rather than decreasing, consistent with

\[ M(< r) \propto r \]

\[ \rho_M \propto r^{-2}. \tag{1.3} \]

This indicates an additional mass component to galaxies which is not luminous and has a density profile \( \rho_M \sim r^{-2} \) which continues out to large radii. This additional, non-luminous mass is referred to as “dark matter”. Further evidence for the dark matter comes from anisotropies in the cosmic microwave background (CMB), which has shown that the dark matter makes up 26.7% of the energy budget of the universe, but ordinary baryonic matter makes up only 4.9% [2]. The case for dark matter has also been made based on large-scale structure of the universe [3, 4] and observations of galaxy clusters [5, 6].

Although dark matter has only been observed through its gravitational interactions, the properties of dark matter are well constrained. Observations of gravitational microlensing have determined that the dark matter is made up of particles with masses \( M_\chi \lesssim 10^{-7} M_\odot \),

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1 Section I is drawn largely from my thesis.
and baryonic clumps of matter cannot explain the majority of the dark matter observed in the Galaxy [7–9]. Additionally, the particles making up the dark matter cannot have ultrarelativistic velocities without underproducing the temperature variations in the CMB, so the dark matter mass must be greater than $M_\chi \gtrsim 10$ eV.

The most popular dark matter model is the weakly interacting massive particle (WIMP), with a dark matter particle of mass $M_\chi \sim 100$ GeV which interacts through the weak force. WIMPs were formed in the early universe in equilibrium with other particles at high temperature. As the universe cooled, WIMPs collided and annihilated into standard model particles. However, once the expansion of the universe exceeded the annihilation rate of the WIMPs, they stopped annihilating and the number density of WIMPs “froze out”, giving the dark matter number density seen today. The so-called “WIMP Miracle” is that for a WIMP with weak-scale cross-section ($\sigma \sim 1$ pb) and a weak-scale mass ($M_\chi \sim 100$ GeV), the relic density of thermal WIMPs is $\Omega_{WIMP} \approx 0.2$, consistent with the energy density for the dark matter. A detailed calculation of the thermal cross-section is found in Appendix A.

Though the WIMP model is usually motivated by Supersymmetry, other theories of beyond-the-Standard-Model physics predict WIMPs as well, including models with extra dimensions. Most of the current models predict dark matter masses from 10 GeV-5 TeV, but WIMPs as heavy as 1000 TeV could exist. The strongest constraint on the upper end of the possible WIMP mass is the unitarity limit. If a particle was formed in the early universe, then it cannot have both an arbitrarily high mass and cross-section. This is because particles with large momentum are less likely to interact with one-another and the velocity of particles in the early universe (when they were in thermal equilibrium) is known. The details of the value of the unitarity limit are shown in Appendix B. If the WIMPs are not produced as thermal relics but are produced non-thermally though decays of heavier particles, or if the cross-section is boosted through Sommerfeld enhancement (discussed in Section III), then these limits are weakened and can be ignored, however [10].

\section{II. CALCULATIONS OF THE DARK MATTER ANNIHILATION FLUX}

\footnote{Section II is drawn largely from my thesis.}
A. Particle Flux from Dark Matter Annihilations

The number of dark matter annihilations per unit volume is \( \rho(\vec{x})^2/(2M_\chi^2)\langle \sigma_A v \rangle \), for dark matter mass \( M_\chi \) and dark matter number density \( \rho/M_\chi \). \( \langle \sigma_A v \rangle \) is the annihilation cross-section times the relative velocity between interacting dark matter particles, averaged over their velocity distribution, and the factor of two comes from symmetry concerns. The number of dark matter collisions per unit time per unit observational area integrated along a particular line-of-sight is then

\[
\frac{dN}{dtdA} = \frac{\langle \sigma_A v \rangle}{2M_\chi^2} \int \frac{r^2drd\Omega}{4\pi r^2} \rho(\vec{x})^2 = \frac{\langle \sigma_A v \rangle}{2M_\chi^2} \int \frac{drd\Omega}{4\pi} \rho(\vec{x})^2 .
\]

(2.1)

Observationally, we detect the number of Standard Model particles per unit energy produced by dark matter annihilations rather than the annihilations themselves. For an annihilation final-state \( f \), the branching fraction \( Br_f \) is the fraction of time a dark matter annihilation will end up in that final state. For a particular Standard Model particle \( i \), \( dN_f^{(i)}/dE \) is the number of particles \( i \) per unit energy produced in final-state \( f \). The details of which final states occur with which branching fractions depends on the precise dark matter model under consideration, though often, for simplicity, a particular dark matter annihilation channel is assumed to dominate and its branching fraction is set to unity. To simulate hadronization and decay of standard model particles into final states, analytical expressions or simulations are used (see section IV). The flux of particles \( i \) per units energy and solid angle coming from a region of dark matter annihilations is given by

\[
\frac{dF_i}{dEd\Omega} = \frac{\langle \sigma_A v \rangle}{2} \sum_f \frac{dN_f^{(i)}}{dE} Br_f \int_0^\infty \frac{dr}{4\pi} \rho(\vec{x})^2 .
\]

(2.2)

B. Dark Matter Distributions

Dark matter in galaxies exists in a roughly spherically symmetric “halo” that approximately falls off as \( \rho(r) \propto r^{-2} \) with the distance from the galactic center. Numerical N-body simulations have been done considering both pure dark matter and dark matter with baryons [11–23]. The most widely adopted dark matter profile is the Navarro-Frenk-White (NFW) profile [21], parameterized as

\[
\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}
\]

(2.3)
with scale radius $r_s \approx 20$ kpc for our Galaxy. The NFW profile is sometimes generalized as

$$\rho(r) = \frac{\rho_s}{(r/r_s)^\gamma(1 + (r/r_s)^\alpha)^{(\beta-\gamma)/\alpha}}$$

(2.4)

to allow for cuspler or more-cored dark matter profiles, with the standard NFW profile having $(\alpha, \beta, \gamma) = (1, 3, 1)$. A somewhat shallower dark matter profile, the Einasto profile, is parameterized as [24, 25]

$$\rho(r) = \rho_s \exp \left[ -\frac{2}{\alpha} \left( \left( \frac{r}{r_s} \right)^\alpha - 1 \right) \right],$$

(2.5)

with scale radius $r_s \approx 20$ kpc and $\alpha = 0.1 - 0.2$ for Milky-Way-type halos [22, 23]. The scale density for these Galactic dark matter profiles, $\rho_s$, is fixed such that the dark matter density at the solar distance $R_\odot \approx 8.5$ kpc is $\rho_\odot \approx 0.39$ GeV cm$^{-3}$ [26–28].

III. CROSS-SECTION AND SUBSTRUCTURE BOOSTS

The dark matter cross-section can be larger than thermal if the dark matter couples to light gauge bosons. The exchange of the bosons by the annihilating dark matter particles can create a resonance and increase the thermal cross-section by orders of magnitude. This is called “Sommerfeld enhancement”, after a similar effect due to photons in electron scattering. Sommerfeld enhancement goes as [29]

$$\epsilon_v \equiv \frac{v}{\alpha},$$

(3.1)

$$\epsilon_\phi \equiv \frac{m_\phi}{\alpha M_\chi},$$

(3.2)

$$S \approx \frac{\pi}{\epsilon_v \cosh \left( \frac{2\pi \epsilon_v}{\pi^2 \epsilon_\phi/6} \right)} \left( \frac{\sinh \left( \frac{2\pi \epsilon_v}{\pi^2 \epsilon_\phi/6} \right)}{\cosh \left( \frac{2\pi \epsilon_v}{\pi^2 \epsilon_\phi/6} \right) - \cos \left( 2\pi \sqrt{\frac{1}{\pi^2 \epsilon_\phi/6} - \frac{\epsilon_v^2}{\left( \frac{\pi^2 \epsilon_\phi/6 \right)^2}} \right)} \right),$$

(3.3)

where $\alpha 1/35$ is the weak coupling, $m_\phi$ is the gauge boson mass, $M_\chi$ is the dark matter mass, $v$ is the relative velocity of the dark matter particles, and $S$ is the enhancement to the cross-section. Because the Sommerfeld enhancement increases at low velocities, and the relative velocity of dark matter in the Galaxy is $\sim 300 \, km \, s^{-1}$, much smaller than the $\sim c/4$ velocities in the early universe, the dark matter cross-section during thermal freeze out can be much less than the cross-section today. The shape of Sommerfeld enhanced cross-section limits have multiple peaks in them, corresponding to the combinations of velocity + dark matter mass + gauge boson mass which lie directly at resonance, not merely near it. Usually,
Sommerfeld enhancement is achieved using a dark-sector light boson. However, for the heavy dark matter masses relevant to HAWC, Standard Model W and Z bosons are light enough to cause this enhancement. Especially for WIMPs which annihilate primarily into $W^+W^-$ or $Z^+Z^-$ states, they are guaranteed to couple to Standard Model gauge bosons and naturally have Sommerfeld enhancement.

The flux can also be boosted due to dark matter substructure. In all numerical simulations of dark matter, most of the dark matter is not in a smooth halo but is found in smaller subhalos, which can have masses as low as $10^{-6} M_\odot$. Because the flux is proportional to the dark matter density squared, and the local density in these subhalos is much larger than from the smooth halo, these substructures can increase the dark matter flux from 2-1000 times, depending on the object. Most of the subhalos are far from the center of the smooth halo [30]. Therefore, in observations of the Galactic center, substructure boosts do not contribute to the flux significantly. For dwarf galaxies, which are actually large, virialized substructures themselves, the substructure boost does not contribute much because the objects are small in extent and do not have that many orders of magnitude difference between the smooth halo mass and the size of the substructure. However, for extragalactic dark matter sources, especially other galaxies and galaxy clusters, the substructure boost can be large.

IV. CALCULATION OF DARK MATTER SPECTRA

To calculate the photon spectrum for a particular WIMP annihilation channel, we use PYTHIA 6.4 to simulate the photon radiation of charged particles as well as decays of particles such as the $\pi^0$ [32]. Specifically, we run PYTHIA to simulate an $e^+e^-$ collision at a center of mass energy of $2M_\chi$ through a $Z'$ to a final state that corresponds to the annihilation products of the dark matter.

We turn off initial state radiation such that all photons only come from the radiation or decay of the dark matter annihilation products. We turn on the decays of particles which are not decayed with the default PYTHIA settings, namely muons, charged pions, and charged kaons. Additionally, we turn on the muon decay channel $\mu^- \to e^-\nu_\mu\bar{\nu}_e\gamma$, with the standard branching fraction of 0.014 [26]. Using a large sample of events for each final state and each value of $M_\chi$, the number of photons in the final state in a given logarithmic energy bin is

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3 Section IV is drawn largely from my paper, Ref. [31]
counted and averaged over the number of events, yielding the average number of photons in that energy bin per annihilation event.

The FORTRAN code I use to run PYTHIA is found in http://private.hawc-observatory.org/svn/hawc/sandbox/harding/dark matter sensitivity paper/spectra. There is also a tarball of the fluxes I used, with energies in TeV, and fluxes in $cm^{-2} s^{-1} TeV^{-1}$, using a cross-section $\langle \sigma v \rangle = 1 \times 10^{-22} cm^3 s^{-1}$. The fluxes have been integrated over the size of the corresponding spatial file. The spatial distribution files (normalized to unity) are found in http://private.hawc-observatory.org/svn/hawc/sandbox/harding/dark_matter_sensitivity_paper/spatial . For large masses, I recompiled PYTHIA with the PYJETS arrays set to 10000 instead of their default 4000, to account for the large number of particles from decays of the annihilation products.

V. CALCULATING DARK MATTER CROSS-SECTION LIMITS

To get the cross-section limits, Brian runs the spectra with the spatial files to get the simulated HAWC significance after 1 day. I convert this to a 5-year significance using $\sqrt{365.25 \times 5}$. Then, because the significance scales with the dark matter cross-section, I adjust the cross-section such that the significance is $2 - \sigma$. Assuming no observation of the source, this should be the cross-section which is excluded at 95% CL.

If we do observe a gamma-ray source in the observation region, the modeling for limits will be tougher. For the Galactic center, we may need to mask out the Galactic plane and any observed point-sources, and for the Virgo cluster, we may need to exclude M87. However, the symmetry of the dark matter halo can be exploited to give essentially only a locally isotropic background, as was done in Ref. [33]. Therefore, we believe that the limits we predict are conservative and do show HAWC’s dark matter sensitivity.

Appendix A: Calculation of the WIMP Cross-Section at Freeze-Out

The annihilation cross-section $\langle \sigma v \rangle$ of WIMPs is set by their thermal freezeout in the early universe. Below, I show how a TeV-scale WIMP annihilating with a weak-scale cross-section gives the expected relic density for dark matter, following Ref. [34]. In the early universe, WIMPs ($\chi$) were in thermal equilibrium with other particles ($\psi$) and interacting

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4 Appendix A is drawn largely from my thesis.
with them via $\psi\bar{\psi} \leftrightarrow \chi\bar{\chi}$ annihilations. The phase-space distribution functions $f$ for the $\psi$ particles are given by

$$f_\psi = \exp \left[ -\frac{E_\psi}{T} \right], \quad f_\bar{\psi} = \exp \left[ -\frac{E_\bar{\psi}}{T} \right],$$

(A1)

assuming there is no chemical potential for the WIMPs. Because $E_\chi + E_{\bar{\chi}} = E_\psi + E_{\bar{\psi}}$,

$$f_\psi f_\bar{\psi} = \exp \left[ -\frac{E_\psi + E_{\bar{\psi}}}{T} \right] = \exp \left[ -\frac{E_\chi + E_{\bar{\chi}}}{T} \right] = f^e q f^e \bar{q}.$$  \hspace{1cm} (A2)

The Boltzmann equation for the WIMPs then reads

$$\frac{d n_\chi}{dt} + 3 H n_\chi = -\langle \sigma_A v | \psi_\chi \rightarrow \psi_\bar{\psi} \rangle \left[ f_\chi f_{\bar{\chi}} - f_\psi f_{\bar{\psi}} \right].$$  \hspace{1cm} (A3)

Summing over all channels, $\sigma_{\chi\bar{\chi} \rightarrow \psi\bar{\psi}} \rightarrow \sigma_A$. To simplify the Boltzmann equation, we now define

$$Y \equiv \frac{n_\chi}{s} = \frac{n_{\bar{\chi}}}{s}$$ \hspace{1cm} (A6)

$$x \equiv \frac{M}{T}$$ \hspace{1cm} (A7)

$$H_M \equiv x^2 H = 1.67 g^1/2 \frac{M^2}{M_{pl}^2}$$ \hspace{1cm} (A8)

where $s$ is the entropy density, $M_{pl}$ is the Planck mass, and $M$ is the mass of the WIMP.

This simplified form of the Boltzmann equation for WIMPs reads

$$\frac{dY}{dx} = -\frac{x s \langle \sigma_A v \rangle}{H_M} \left( Y^2 - Y^2_{eq} \right).$$  \hspace{1cm} (A9)

When $\Gamma_A \equiv n_{eq} \langle \sigma_A v \rangle \ll H$ we have $dY \ll Y_{eq} d \ln(x)$, so $Y$ changing little with respect to its equilibrium value and is said to have undergone “freeze-out”. The value of $x$ where $\Gamma_A \approx H$ is labeled $x_f$.

The annihilation cross-section for WIMPs should go as $\langle \sigma_A v \rangle \propto v^{2l}$ with $l = 0$ for s-wave annihilation, $l = 1$ for p-wave annihilation, and so on. Because WIMPs freeze out late ($x_f > 3$) they are non-relativistic and therefore s-wave ($l = 0$) annihilation should dominate. Conveniently, this means that $\langle \sigma_A v \rangle$ is a constant, independent of temperature. The Boltzmann equation for s-wave annihilation is

$$\frac{dY}{dx} = -\lambda x^{-2} \left( Y^2 - Y^2_{eq} \right).$$  \hspace{1cm} (A10)
with

\[ Y_{eq} = \frac{n_{\chi}^e}{s} = \left[ g \left( \frac{MT}{2\pi} \right)^{3/2} \exp \left( -\frac{M}{T} \right) \right] \left[ \frac{45}{2\pi^2 g_{ss} T^3} \right] = 0.145 \frac{g}{g_{ss}^{1/2}} x^3 e^{-x} \quad (A11) \]

\[ \lambda = \frac{x s \langle \sigma_A v \rangle}{H} x^2 = \left[ \frac{2\pi^2}{45} g_{ss} T^3 x^3 \langle \sigma_A v \rangle \right] \left[ \frac{M_{pl}}{1.67 g_{s}^{1/2} M^2} \right] \]

\[ = 0.264 \frac{g_{ss}}{g_{s}^{1/2}} M_{pl} M \langle \sigma_A v \rangle \quad (A12) \]

where we have used the entropy density

\[ s = \frac{2\pi^2}{45} g_{ss} T^3 . \quad (A13) \]

For \( x \gg x_f \), \( Y \) is frozen out and remains fairly constant while \( Y_{eq} \) dies exponentially, so \( Y \gg Y_{eq} \) is a good approximation. We find an approximate solution to Eq. A10 for large \( x \) by integrating from \( x_f \) to \( \infty \) to get

\[ Y_{\infty} \approx \frac{x_f}{\lambda} \quad (A14) \]

\[ x_f \approx \ln \left[ 0.038 \frac{g}{g_{s}^{1/2}} M_{pl} M \langle \sigma_A v \rangle \right] - \frac{1}{2} \ln \left\{ \ln \left[ 0.038 \frac{g}{g_{s}^{1/2}} M_{pl} M \langle \sigma_A v \rangle \right] \right\} . \quad (A15) \]

The present number density of WIMPs is

\[ n_{\chi 0} = s_0 Y_{\infty} \approx (2970 \text{cm}^{-3}) Y_{\infty} \quad (A16) \]

\[ \Omega_{\chi} h^2 = \frac{M n_{\chi 0}}{\rho_c} \quad (A17) \]

\[ \rho_c = \frac{3H_0^2}{8\pi G} . \quad (A18) \]

Specifically, we will choose a WIMP with mass \( M \), annihilation cross-section \( \langle \sigma_A v \rangle = a \cdot (3 \times 10^{-26} \text{cm}^3 \text{s}^{-1}) \), and two degrees of freedom \((g = 2)\) that freezes out at \( T \sim 100 \text{MeV} \) \((g_s = g_{ss} \approx 60)\). For these parameters,

\[ x_f \approx 19.6 + \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] - \frac{1}{2} \ln \left\{ 19.6 + \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] \right\} \]

\[ \approx 18 + \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] \quad (A19) \]

\[ Y_{\infty} \approx \frac{1.56 \times 10^{-11}}{a (M/\text{GeV})} \left( 18 + \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] \right) \]

\[ \approx \frac{2.81 \times 10^{-10}}{a (M/\text{GeV})} \left( 1 + 0.0556 \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] \right) \quad (A20) \]

\[ \Omega_{\chi} h^2 \approx 4.41 \times 10^{-3} \frac{x_f}{a} \approx 0.0794 \frac{a}{a} \left( 1 + 0.0556 \ln \left[ a \left( \frac{M}{\text{GeV}} \right) \right] \right) . \quad (A21) \]
TABLE I. WIMP annihilation cross-section vs WIMP mass, for a standard thermal WIMP. The cross-sections have been calculated to leading logarithmic order.

<table>
<thead>
<tr>
<th>$M$ [GeV]</th>
<th>$\langle \sigma v \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.08 \times 10^{-26}$</td>
</tr>
<tr>
<td>3</td>
<td>$2.22 \times 10^{-26}$</td>
</tr>
<tr>
<td>10</td>
<td>$2.37 \times 10^{-26}$</td>
</tr>
<tr>
<td>30</td>
<td>$2.50 \times 10^{-26}$</td>
</tr>
<tr>
<td>100</td>
<td>$2.65 \times 10^{-26}$</td>
</tr>
<tr>
<td>300</td>
<td>$2.79 \times 10^{-26}$</td>
</tr>
<tr>
<td>1000</td>
<td>$2.93 \times 10^{-26}$</td>
</tr>
<tr>
<td>3000</td>
<td>$3.07 \times 10^{-26}$</td>
</tr>
<tr>
<td>$1 \times 10^4$</td>
<td>$3.22 \times 10^{-26}$</td>
</tr>
<tr>
<td>$3 \times 10^4$</td>
<td>$3.35 \times 10^{-26}$</td>
</tr>
<tr>
<td>$1 \times 10^5$</td>
<td>$3.50 \times 10^{-26}$</td>
</tr>
</tbody>
</table>

The current best-fit value for the dark matter fraction is $\Omega_{CDM}h^2 = 0.1122$ from the WMAP collaboration [35]. Neglecting the logarithmic term, this gives $\langle \sigma_A v \rangle \approx 2.1 \times 10^{-26}$ cm$^3$s$^{-1}$, which is close to the canonical value of $3 \times 10^{-26}$ cm$^3$s$^{-1}$, corresponding to a weak-scale WIMP annihilation cross-section $\sigma_A \approx 1$ pb. Including the logarithmic term, the cross-section is dependent on the WIMP mass, though only by $\sim 50$ percent over four orders of magnitude in $M$. The cross-section for a range of masses is given in Table I. The solution, labeled $\langle \sigma_A v \rangle$, varies by only $\sim 50$ percent over four orders of magnitude in $M$, so the standard assumption that $\langle \sigma_A v \rangle = 3 \times 10^{-26}$ cm$^3$s$^{-1}$ regardless of mass is very well-motivated.
Appendix B: Unitarity Bounds on the WIMP Cross-Section

For multi-TeV WIMPs, the unitarity of the scattering matrix \( S \) provides an upper limit on the dark matter annihilation cross-section, which I show here following Refs. \[36, 37\]. Unitarity of \( S \) provides that \( S^\dagger S = 1 \). Therefore,

\[
(1 - S)(1 - S)^\dagger = (1 - S^\dagger) + (1 - S) \quad .
\] (B1)

For the scattering from one state \( |i\rangle \) to an identical state \( |i\rangle \), then unitarity gives

\[
2 \text{Re}[\langle i|(1 - S)|i\rangle] = \langle i|(1 - S) + (1 - S^\dagger)|i\rangle \quad \quad \quad (B2)
\]

\[
= \langle i|(1 - S)(1 - S)^\dagger|i\rangle \quad \quad \quad (B3)
\]

\[
= \int d\gamma \langle i|(1 - S)|\gamma\rangle \langle \gamma|(1 - S)^\dagger|i\rangle \quad \quad \quad (B4)
\]

where \( d\gamma \) covers the complete set of states \( \gamma \). By definition, the scattering amplitude \( A_{\beta\alpha} \) is

\[
\langle \beta|(1 - S)|\alpha\rangle \equiv -i(2\pi)^4\delta^4\left(\sum p_\beta - \sum p_\alpha\right)A_{\beta\alpha} \quad .
\] (B5)

Inserting Eq. B5 into Eq. B4 gives

\[
2 \text{Re}[−i(2\pi)^4\delta^4(p_i - p_\gamma)A_{i\gamma}] = \int d\gamma(−i(2\pi)^4\delta^4(p_i - p_\gamma)A_{i\gamma}) \times
\]

\[
\times \{i(2\pi)^4\delta^4(p_i - p_\gamma)A_{i\gamma}^*\} \quad \quad \quad (B6)
\]

\[
2 \text{Im}[(2\pi)^4\delta^4(0)A_{i\gamma}] = \int d\gamma(2\pi)^8\delta^4(0)\delta^4(p_i - p_\gamma)|A_{i\gamma}|^2 \quad \quad \quad (B7)
\]

\[
2 \text{Im}[A_{i\gamma}] = \int d\gamma(2\pi)^4\delta^4(p_i - p_\gamma)|A_{i\gamma}|^2 \quad . \quad \quad (B8)
\]

For any two particles scattering in the center-of-mass frame, the differential cross-section is \[38\]

\[
d\sigma(i \to \gamma) = \frac{(2\pi)^4|A_{\gamma i}|^2}{4(E_1 + E_2)|\vec{p}_1|}\delta^4\left(\sum p_i - \sum p_\gamma\right) d\gamma \quad , \quad \quad \quad \quad (B9)
\]

where the center-of-mass energies of the initial two particles are \( E_1 \) and \( E_2 \), the center-of-mass momentum of one of the initial particles is \( \vec{p}_1 \), and all final states’ phase-space are covered by

\[
d\gamma = \frac{d^3p_1}{(2\pi)^32E_1} \frac{d^3p_2}{(2\pi)^32E_2} \ldots \quad . \quad \quad (B10)
\]
Inserting Eq. B8 into Eq. B9, we get the optical theorem

$$\sigma_{\text{tot}} = \int d\gamma \frac{d\sigma}{d\gamma}(i \rightarrow \gamma) = \frac{\text{Im}[A_{ii}]}{2(E_1 + E_2)|\vec{p}_1^0|}.$$  \hspace{1cm} (B11)

Our next goal is to calculate the total cross-section and inelastic cross-sections for 2-to-2 particle scattering, in the center-of-mass frame. Let \( |i\rangle = |p_1, p_2, s_{1z}, s_{2z}\rangle \) and \( |\gamma\rangle = |p'_1, p'_2, s'_{1z}, s'_{2z}\rangle \) be the initial and final states, with 4-momenta and spins, in 2-to-2 particle scattering, respectively. Starting with Eq. B5, we sum over final particle momenta and insert a set of complete angular momentum states around the scattering matrix. The sum over final particle momenta yields the expression

$$\sum_{j,j_z} \sum_{l,l_z} \sum_{s,s_z} \sum_{l',l'_z} \sum_{s',s'_{z}} \langle j j_z | jj_z' \rangle \langle l l_z | ll_z' \rangle \langle s s_z | s s_z' \rangle \langle l' l'_z | l' l'_z' \rangle \langle s' s'_{z} | s' s'_{z}' \rangle = \frac{\delta(E'_1 + E'_2 - E_1 - E_2)}{E'_1 E'_2} \frac{|\vec{p}_1|^2 d|\vec{p}_1| d\Omega}{4(2\pi)^2}$$

$$= \frac{1}{4(2\pi)^2} \frac{|\vec{p}_1|^2 d\Omega}{E'_1 + E'_2} \delta(W - E'_1 - E'_2) dW$$

(B12)

where

$$W = E_1 + E_2 = E'_1 + E'_2 = \sqrt{M_1^2 + |\vec{p}_1|^2} + \sqrt{M_2'^2 + |\vec{p}_1'|^2}$$  \hspace{1cm} (B13)

$$dW = d|\vec{p}_1'| \left( \frac{|\vec{p}_1|^2}{E'_1} + \frac{|\vec{p}_1|^2}{E'_2} \right) = d|\vec{p}_1||\vec{p}_1'| \frac{E'_1 + E'_2}{E'_1 E'_2}.$$  \hspace{1cm} (B14)

Around the scattering matrix, we also insert the complete set of angular momentum states

$$\sum_{j,j_z} \sum_{l,l_z} \sum_{s,s_z} \sum_{l',l'_z} \sum_{s',s'_{z}} \langle j j_z | jj_z' \rangle \langle l l_z | ll_z' \rangle \langle s s_z | s s_z' \rangle \langle l' l'_z | l' l'_z' \rangle \langle s' s'_{z} | s' s'_{z}' \rangle$$

(B15)

to get the complete scattering amplitude

$$A_{ij} = 4i(2\pi)^2 \frac{E_1 + E_2}{|\vec{p}_1|} \sum_{j,j_z} \sum_{l,l_z} \sum_{s,s_z} \sum_{l',l'_z} \langle l' s'| (1 - S) | l s \rangle \times$$

$$\times \sum_{s,s_z} \langle s s_z | s s_z' \rangle \langle s s_z | s s_z' \rangle^* \times$$

$$\times \sum_{l,l_z} \langle j j_z | l l_z' \rangle \langle j j_z | l l_z' \rangle^* \langle \vec{p}_1 | l l_z \rangle \langle \vec{p}_1 | l l_z \rangle^*.$$  \hspace{1cm} (B16)

(B17)

If we let \( \hat{p}_1 = \hat{z} \), average over initial spins, and set \( |\gamma\rangle = |i\rangle \), then

$$A_{ii} = \frac{4\pi i}{(2s_1 + 1)(2s_2 + 1)} \frac{E_1 + E_2}{|\vec{p}_1|} \sum_j (2j + 1) \sum_{l,s} \langle j s | (1 - S) | j s \rangle.$$  \hspace{1cm} (B18)
Finally, inserting Eq. B18 into the optical theorem, Eq. B11, we get the total 2-body cross-section
\[
\sigma_{\text{tot}} = \frac{2\pi}{(2s_1+1)(2s_2+1)|\hat{p}_1|^2} \sum_j (2j+1) \sum_{l,s} \text{Re}[\langle l\ s|(1-S)|l\ s\rangle].
\]  
(B19)

The elastic cross-section is the cross-section for $\chi\chi \to \chi\chi$ or $\chi\bar{\chi} \to \chi\bar{\chi}$. The cross-section for this interaction is given by [38]
\[
d\sigma_{\text{el}} = \frac{|A_{fi}|^2}{4(E_1 + E_2)|\hat{p}_1|^2} (2\pi)^4 \delta^4 \left( \sum p_f - \sum p_i \right) \frac{d^3p_1'}{(2\pi)^3 E_1} \frac{d^3p_2'}{(2\pi)^3 E_2}.  
\]  
(B20)

From Eq. B16, averaging over initial spins, and letting $\hat{p}_1 = \hat{z}$, we find that
\[
|A_{fi}|^2 = \frac{64\pi^3}{(2s_1+1)(2s_2+1)} \frac{(E_1 + E_2)^2}{|\hat{p}_1|^2} \sum_{j,j,s,s'} \sum_{l,l',s,s'} |\langle l' s'|(1-S)|l\ s\rangle|^2
\]  
(B21)

where we have used $\int d\Omega' |Y_l' s'|^2 = \delta_{l' l} \delta_{s' s} = 1$ for the outgoing particle directions $\Omega'$. From Eqs. B12, B20, and B21, the total elastic cross-section is
\[
\sigma_{\text{el}} = \frac{1}{(2s_1+1)(2s_2+1)|\hat{p}_1|^2} \sum_j (2j+1) \sum_{l,l',s,s'} |\langle l' s'|(1-S)|l\ s\rangle|^2.  
\]  
(B22)

The inelastic cross-section ($\chi\chi \to \chi\chi$ or $\chi\bar{\chi} \to \chi\bar{\chi}$) is given by $\sigma_{\text{tot}} - \sigma_{\text{el}}$
\[
\sigma_{\text{inel}} = \frac{\pi}{(2s_1+1)(2s_2+1)|\hat{p}_1|^2} \sum_j (2j+1) \times \left\{ \sum_{l,s} 2\text{Re}[\langle l\ s|(1-S)|l\ s\rangle] - \sum_{l',s,s'} |\langle l' s'|(1-S)|l\ s\rangle|^2 \right\}  
\]  
(B23)

\[
= \frac{\pi}{(2s_1+1)(2s_2+1)|\hat{p}_1|^2} \sum_j (2j+1) \times \sum_{l,s} \left\{ 1 - |\langle l\ s|S|l\ s\rangle|^2 - \sum_{l\neq l',s\neq s'} |\langle l' s'|(1-S)|l\ s\rangle|^2 \right\}  
\]  
(B24)

where we have used
\[
|\langle l\ s|(1-S)|l\ s\rangle|^2 = (\langle l\ s|(1-S)|l\ s\rangle)^*(\langle l\ s|(1-S)|l\ s\rangle) = 1 - 2\text{Re}[\langle l\ s|S|l\ s\rangle] + |\langle l\ s|S|l\ s\rangle|^2.  
\]  
(B25)

To place constraints on the cross-sections, we will place limits on the S-matrix elements in Eqs. B19 and B24. Trivially,
\[
\text{Re}[\langle \alpha|(1-S)|\alpha\rangle] = 1 - \text{Re}[\langle \alpha|S|\alpha\rangle] \leq 1 + |\text{Re}[\langle \alpha|S|\alpha\rangle]| \leq 1 + |\langle \alpha|S|\alpha\rangle|.  
\]  
(B26)
Furthermore,

\[ |\langle \alpha | S | \alpha \rangle|^2 = \langle \alpha | S^\dagger | \alpha \rangle \langle \alpha | S | \alpha \rangle \leq \int d\gamma \langle \alpha | S^\dagger | \gamma \rangle \langle \gamma | S | \alpha \rangle = \langle \alpha | S^\dagger S | \alpha \rangle = 1 \] (B27)

because \(|\alpha\rangle\) is one of the states summed over in \(|\gamma\rangle\). Therefore, we can place an upper limit on the total cross-section through:

\[ \text{Re}[\langle \alpha | (1 - S) | \alpha \rangle] \leq 2 \] (B28)

\[ \sigma_{\text{tot}} \leq \frac{4\pi}{(2s_1 + 1)(2s_2 + 1)|\vec{p}_1|^2} \sum_{j,l,s} (2j + 1) . \] (B29)

An upper bound on the inelastic cross-section can be found as well, using

\[ \langle l s | S^\dagger | l s \rangle \geq 0 \] (B30)

\[ \langle l' s' | (1 - S) | l s \rangle \geq 0 \] (B31)

\[ 1 - \langle l s | S | l s \rangle^2 - \langle l' s' | (1 - S) | l s \rangle^2 \leq 1 \] (B32)

\[ \sigma_{\text{inel}} \leq \frac{\pi}{(2s_1 + 1)(2s_2 + 1)|\vec{p}_1|^2} \sum_{j,l,s} (2j + 1) . \] (B33)

For thermalized WIMPs annihilating non-relativistically with relative velocity \(v_{\text{rel}}\) the momentum is related to the relative velocity and the WIMP mass by

\[ |\vec{p}_1|^2 = E_1^2 - M^2 = M^2 \frac{v_1^2}{1 - v_1^2} = \frac{M^2}{4} \frac{v_{\text{rel}}^2}{1 - v_{\text{rel}}^2/4} \approx \frac{M^2}{4} \frac{v_{\text{rel}}^2}{4} \] (B34)

because \(\vec{v}_{\text{rel}} = \vec{v}_1 - \vec{v}_2 = 2\vec{v}_1\). Therefore, for WIMPs with relative velocity \(v_{\text{rel}}\) and mass \(M_\chi\) the cross-section is limited by

\[ \sigma_{\text{tot}} \leq \frac{16\pi}{(M_\chi v_{\text{rel}})^2} = 1.76 \times 10^{-25} \text{ cm}^2 \left( \frac{\text{TeV}}{M_\chi} \right)^2 \left( \frac{100 \text{ km s}^{-1}}{v_{\text{rel}}} \right)^2 \] (B35)

\[ \sigma_{\text{inel}} v_{\text{rel}} \leq \frac{4\pi}{M^2_\chi v_{\text{rel}}} = 4.40 \times 10^{-19} \text{ cm}^3 \text{s}^{-1} \left( \frac{\text{TeV}}{M_\chi} \right)^2 \left( \frac{100 \text{ km s}^{-1}}{v_{\text{rel}}} \right) \] (B36)

for an s-wave annihilation (as is expected for non-relativistic WIMP annihilations), with \(v_{\text{rel}} \sim 300 \text{ km s}^{-1}\) in the Milky Way, \(v_{\text{rel}} \sim 1000 \text{ km s}^{-1}\) in clusters, and \(v_{\text{rel}} \sim 1.50 \times 10^5 \text{ km s}^{-1}\) at freeze-out, assuming those WIMPs are thermalized [36, 37].

[2] Planck Collaboration Collaboration, P. Ade et al., *Planck 2013 results. XVI.*


[arXiv:1205.4033].


